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Periodic Solutions of Kadomtsev–Petviashvili

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This paper establishes the existence of global solutions of the nonlinear equation of Kadomtsev–Petviashvili

$$u_t + uu_x + u_{xxx} = D^{-1}u_{yy}, \tag{1}$$

which are periodic in x and y and D^{-1} denotes the primitive in x.

Zakharov [1], Chen [2], and Fokas [3] have established the existence of an infinite sequence of integrals of motion $F_n(u)$, $n \geqslant 1$, and shown that the Poisson bracket of $F_n(u)$ with $F_{n'}(u)$ vanishes for all n and n' and smooth periodic u. Except for the special solutions of Novikov and Krichever [4], it was not known if there were general global solutions of (1). We show that solutions of (1) exist for all time, are uniquely determined by their initial values, and that these values can be prescribed almost arbitrarily.

To formulate this result, we introduce the following notation. The norm in the space L_p is denoted by $|v|_p$, in L_2 by $\|v\| = |v|_2$, and $\max|v| = \max_{x,y}|v(x, y)|$. Let $D^{-1}v = \int_0^x v$ and V_m signify the set of periodic functions with

$$\|v\|_{V_m} = \sum_{n=0}^{m} \sum_{l=0}^{n} \|D_x^{n-l}(D_x^{-1}D_y)^l v\| < \infty \qquad \text{for} \quad m = 0, 1, 2, \ldots$$

with $\int_0^1 v(x, y)\, dx = 0$.

A global solution of (1) is a functon $u(x, y, t)$ such that $\|u(t)\|_{V_m}$, $m \geqslant 3$ is bounded for all t and u satisfies (1).

THEOREM. *If the initial data g is an element of V_3 with small $\|g\|$, then there exists a unique global solution $u(x, y, t)$ to (1) from V_3 with $\|u(t)\|_{V_3} \leqslant c$ where c depends only on $\|g\|_{V_3}$.* The solution $u(t)$ in V_2 is Lipchitz continuous in $g \in V_3$ locally uniformly in t.

This result is established in the next section. We give the apriori bound

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for the solution $u(x, y, t)$ of (1) in Lemma 1. This depends on the integrals $F_n(u)$ which may be computed from Chen [2] and Fokas [3]. For u in V_m, the first few are:

$$F_1(u) = \int_0^1 \int_0^1 u^2\,dx\,dy, \qquad F_2(u) = \int_0^1 \int_0^1 uD^{-1}u_y\,dx\,dy,$$

$$F_3(u) = \int_0^1 \int_0^1 u_x^2 - \tfrac{1}{3}u^3 + (D^{-1}u_y)^2\,dx\,dy,$$

$$\begin{aligned} F_5(u) = \int_0^1 \int_0^1 &\tfrac{3}{2}u_{xx}^2 + 5u_y^2 + \tfrac{5}{6}(D^{-2}u_{yy})^2 \\ &+ \tfrac{5}{9}D^{-1}uu_yD^{-1}u_y - \tfrac{5}{9}(D^{-1}u_y)^2\,u \\ &- \tfrac{5}{6}u^2D^{-2}u_{yy} - \tfrac{5}{2}u_x^2u + \tfrac{5}{24}u^4\,dx\,dy. \end{aligned}$$

The remaining functionals may be computed from [2] and [3]. For n odd and integer r, each term in $F_n(u)$ has weight $n+1$, the weight of a term being defined as the sum of the weights of its factors, so that the weight of $D^{-r_1}D_x^{r_2}D_y^{r_3}u$ is $1+2r_3+1r_2-1r_1$. If $u(x, y, t)$ is a solution from V_3 of (1) and $\|g\|$ is small then

$$\int_0^1 \int_0^1 u_x^2 + (D^{-1}u_y)^2 - \tfrac{1}{3}u^3\,dx\,dy$$

is positive. Then the constants $F_n(u)$ may be used to show that $\|u(t)\|_{V_3}$ is controlled by a constant depending only on $\|g\|_{V_3}$. The sense in which the solution $u(x, y, t)$ of (1) depends on its initial values is established in Lemma 3. Finally we give an existence and uniqueness argument which leads to the proof of the theorem.

The result may be extended, by using the remaining functionals $F_n(u)$, $n \geqslant 9$ in [2] to show for small $\|g\|$ that $\|u(t)\|_{V_m}$, $m \geqslant 4$ is bounded by a constant which depends only on $\|g\|_{V_m}$. The result follows by taking D^{-1} to be the primitive with $D^{-1}v$ periodic and $\int_0^1 D^{-1}v\,dx = 0$.

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We first establish an apriori bound for the solution $u(x, y, t)$ of (1).

LEMMA 1. *For all g in V_3 with small $\|g\|$, the solution of* (1) *satisfies $\|u(t)\|_{V_3} \leqslant c$, where the constant depends only on $\|g\|_{V_3}$.*

Proof. The proof falls into four parts.

Item 1. $F_1(u)$ is an integral so

$$\int_0^1 \int_0^1 u^2 \, dx \, dy = \int_0^1 \int_0^1 g^2 \, dx \, dy.$$

Item 2. Write (1) in the form

$$u_t = D(D^{-2}u_{yy} - \tfrac{1}{2}u^2 - u_{xx}).$$

Multiply by $D^{-2}u_{yy} - \frac{1}{2}u^2 - u_{xx}$, integrate in x and y, integrate by parts, and then integrate in t to confirm

$$\int_0^1 \int_0^1 (\tfrac{1}{2}(D^{-1}u_y)^2 - \tfrac{1}{6}u^3 + \tfrac{1}{2}u_x^2) \, dx \, dy$$
$$= \int_0^1 \int_0^1 (\tfrac{1}{2}(D^{-1}g_y)^2 - \tfrac{1}{6}g^3 + \tfrac{1}{2}g_x^2) \, dx \, dy. \tag{2}$$

The estimate

$$\int_0^1 \int_0^1 u^3 \, dx \, dy \leqslant 2\sqrt{3} \, \|u\| \, \|u_x\|^{3/2} \, \|D^{-1}u_y\|^{1/2} \tag{3}$$

is used to obtain the apriori bounds for $\|u_x\|$ and $\|D^{-1}u_y\|$, as follows. Begin with

$$\left|\int_0^1 \int_0^1 u^3 \, dx \, dy\right| = 2 \left|\int_0^1 \int_0^1 u u_x D^{-1}u \, dx \, dy\right| \leqslant 2 \, \|u\| \, \|u_x\| \max |D^{-1}u|.$$

Estimate $D^{-1}u = \int^x u \, dx$ to obtain $|D^{-1}u| \leqslant \{\int_0^1 u^2 \, dx\}^{1/2}$. Now write

$$u^2(y) = 2 \int_{y_0}^y u u_y \, dy + u^2(y_0),$$

integrate in x and then by parts in x to confirm

$$\int_0^1 u^2(y) \, dx = -2 \int_0^1 \int_{y_0}^y u_x D^{-1}u_y \, dx \, dy + \int_0^1 u^2(y_0) \, dx.$$

By the mean value theorem,

$$\int_0^1 u^2 \, dx \leqslant 2 \, \|u_x\| \, \|D^{-1}u_y\| + \|u\|^2. \tag{4}$$

Combine with

$$\int_0^1\int_0^1 u^2\,dx\,dy = \left|\int_0^1\int_0^1 u_x D^{-1}u\,dx\,dy\right|$$
$$\leqslant \int\left(\max_y |D^{-1}u| \int |u_x|\,dy\right)dx$$
$$\leqslant \left(\int \max_y |D^{-1}u|^2\,dx\right)^{1/2}\left(\int\left(\int |u_x|\,dy\right)^2 dx\right)^{1/2}$$
$$\leqslant \left(\int\left(\int |D^{-1}u_y|\,dy\right)^2 dx\right)^{1/2} \|u_x\|$$
$$\leqslant \|D^{-1}u_y\|\,\|u_x\| \tag{5}$$

to find

$$|D^{-1}u(x,y)| \leqslant \sqrt{3}\,\|D^{-1}u_y\|^{1/2}\,\|u_x\|^{1/2} \tag{6}$$

and

$$\int_0^1\int_0^1 u^3\,dx\,dy \leqslant 2\sqrt{3}\,\|u\|\,\|u_x\|^{3/2}\,\|D^{-1}u_y\|^{1/2}.$$

Apply (3) to (2) to obtain

$$\tfrac{1}{2}\|u_x\|^2 + \tfrac{1}{2}\|D^{-1}u_y\|^2 \leqslant c + 2\sqrt{3}\,\|u\|\,\|u_x\|^{3/2}\,\|D^{-1}u_y\|^{1/2}.$$

By Item 1 and Young's inequality,

$$\tfrac{1}{2}\|u_x\|^2 + \tfrac{1}{2}\|D^{-1}u_y\|^2 \leqslant c + c_\varepsilon\,\|g\|^4\,\|D^{-1}u_y\|^2 + \varepsilon\,\|u_x\|^2$$

for arbitrary ε. Select ε and then $\|g\|$ small enough to guarantee

$$\|u_x\|^2 + \|D^{-1}u_y\|^2 \leqslant c$$

where c depends on $\|g_x\|$, and $\|D^{-1}g_y\|$.

Item 3. Up to inessential contants, the quadratic terms in $F_5(u)$ are

$$\int_0^1\int_0^1 (u_{xx}^2 + u_y^2 + (D^{-2}u_{yy})^2 + u^4)\,dx\,dy.$$

The inequality

$$\left|\int_0^1\int_0^1 u^4\,dx\,dy\right| = 3\left|\int_0^1\int_0^1 u^2 u_x D^{-1}u\,dx\,dy\right|$$
$$\leqslant 3\max|D^{-1}u|\,\|u_x\|\,|u|_4^2$$
$$\leqslant 3\sqrt{3}\,\|u_x\|^{1/2}\,\|D^{-1}u_y\|^{1/2}\,\|u_x\|\,|u|_4^2$$

which leads to

$$|u|_4 \leqslant 3 \, \|u_x\|^{3/4} \, \|D^{-1}u_y\|^{1/4} \tag{7}$$

is used to estimate the remaining terms in $F_5(u)$ as follows:

$$\left|\int_0^1 \int_0^1 u^2 D^{-2}u_{yy}\, dx\, dy\right| \leqslant c\, \|D^{-2}u_{yy}\| \leqslant \tfrac{1}{2} \|D^{-2}u_{yy}\|^2 + c$$

$$\left|\int_0^1 \int_0^1 (D^{-1}u_y)^2\, u\, dx\, dy\right| \leqslant \tfrac{1}{4} \|u_{xx}\|^2 + \tfrac{1}{6} \|u_y\|^2 + c$$

$$\left|\int_0^1 \int_0^1 u_y D^{-1}u_y D^{-1}u\, dx\, dy\right| \leqslant \tfrac{1}{6} \|u_y\|^2 + c$$

$$\left|\int_0^1 \int_0^1 uu_x^2\, dx\, dy\right| \leqslant \tfrac{1}{4} \|u_{xx}\|^2 + \tfrac{1}{6} \|u_y\|^2 + c.$$

The details are routine. Use is made of (6), (7), and Young's inequality. Combining, we obtain

$$\|u_{xx}\|^2 + \|u_y\|^2 + \|D^{-2}u_{yy}\|^2 \leqslant c,$$

where c depends on $\|g_x\|$, $\|D^{-1}g_y\|$, $\|g_y\|$, $\|g_{xx}\|$.

Item 4. Compute $F_7(u)$ using the recursion formula in [3] to confirm that up to inessential constants the quadratic terms are

$$\int_0^1 \int_0^1 u_{xxx}^2 + u_{xy}^2 + (D^{-1}u_{yy})^2 + (D^{-3}u_{yyy})^2\, dx\, dy.$$

The terms of order three in u with weight eight involve the product of any two of u_{xx}, $D^{-1}u_y$, u_x, u_y, or u with $D^{-3}u_{yyy}$, $D^{-1}u_{yy}$, u_{xy}, or u_{xxx}. Combine with terms that are of higher order in u, estimate and apply (6), (7), and previous bounds to obtain

$$\|u_{xxx}\|^2 + \|u_{xy}\|^2 + \|D^{-1}u_{yy}\|^2 + \|D^{-3}u_{yyy}\|^2 \leqslant c,$$

where c depends on $\|g_{xxx}\|$, $\|g_{xy}\|$, $\|D^{-1}g_{yy}\|$, and $\|D^{-3}g_{yyy}\|$. This completes the proof of Lemma 1.

LEMMA 2. *The initial value problem for* (1) *has no more than one solution* $u(t)$ *from* V_3.

Proof. Let u and v be solutions of (1) with common initial data g. With $w = u - v$, (1) becomes

$$w_t = D^{-1}w_{yy} - wu_y - w_x v - w_{xxx}.$$

Multiply by w and integrate in x and y to obtain

$$\frac{1}{2}\int_0^1\int_0^1 \frac{\partial}{\partial t} w^2\,dx\,dy=\int_0^1\int_0^1 (\tfrac{1}{2}v_x-u_x)\,w^2\,dx\,dy$$

after integrating by parts and using the fact that u and v belong to V_3 for each t. With $\max|v_x|$ and $\max|u_x|$ bounded,

$$\frac{1}{2}\frac{\partial}{\partial t}\int_0^1\int_0^1 w^2\,dx\,dy\leqslant c\int_0^1\int_0^1 w^2\,dx\,dy$$

Gronwall's inequality completes the proof.

LEMMA 3. *Let $T>0$ be finite and let u and v be two solutins of* (1) *with initial data g and h in V_3. Then*

$$\max_{|t|\leqslant T}\|u(t)-v(t)\|_{V_2}\leqslant c\,\|g-h\|_{V_3}, \tag{8}$$

where c depends on T and g and h in V_3.

Proof. $w=u-v$ satisfies

$$w_t=D^{-1}w_{yy}-ww_x-(vw)_x-w_{xxx}. \tag{9}$$

Item 1. Multiply (9) by w, integrate in x and y and then by parts to obtain

$$\frac{\partial}{\partial t}\int_0^1\int_0^1 w^2\,dx\,dy=-\int_0^1\int_0^1\left(www_x+\frac{w^2}{2}v_x\right)dx\,dy.$$

Integrate in t and use standard estimates to prove that

$$\begin{aligned}\|w(t)\|^2&\leqslant\|w(0)\|^2+(\max|u_x|+2\max|v_x|)\int_0^t\|w\|^2\,ds\\&\leqslant\|w(0)\|^2+c\int_0^t\|w(s)\|^2\,ds\end{aligned}$$

where c depends on $\max|u_x|$ and $\max|v_x|$.

Item 2. Next differentiate (9) in x and multiply by w_x; then multiply (9) by $D^{-2}w_{yy}$. Combine, integrate in x and y to obtain

$$\begin{aligned}&\frac{1}{2}\frac{\partial}{\partial t}\int_0^1\int_0^1 w_x^2+(D^{-1}w_y)^2\,dx\,dy\\&\quad=-\int_0^1\int_0^1((ww)_x\,w_{xx}+(vw)_x\,w_{xx}))\,dx\,dy\\&\qquad-\int_0^1\int_0^1(D^{-2}w_{yy}ww_x+D^{-2}w_{yy}(vw)_x)\,dx\,dy.\end{aligned}$$

Integrate in t and estimate each term using (6) and (7). The estimates are similar to the following:

$$\left|\int_0^1\int_0^1 w_{xx}v_x w\,dx\,dy\right| \leqslant \left|\int_0^1\int_0^1 w_x^2 v_x\,dx\,dy\right| + \left|\int_0^1\int_0^1 w_x v_{xx} w\,dx\,dy\right|$$

$$\leqslant \max|v_x|\,\|w_x\|^2 + \tfrac{1}{2}\left|\int_0^1\int_0^1 w^2 v_{xxx}\,dx\,dy\right|$$

$$\leqslant c\,\|w_x\|^2 + \tfrac{1}{2}\|v_{xxx}\|\,|w|_4^2$$

$$\leqslant c(\|w_x\|^2 + \|D^{-1}w_y\|^2)$$

$$\left|\int_0^1\int_0^1 D^{-2}w_{yy}ww_x\,dx\,dy\right| = \tfrac{1}{2}\left|\int_0^1\int_0^1 (D^{-1}w_y)^2 w_x\,dx\,dy\right|$$

$$\leqslant c\,\|D^{-1}w_y\|^2$$

$$\left|\int_0^1\int_0^1 D^{-1}w_y v_y w\,dx\,dy\right| \leqslant \tfrac{1}{2}\left|\int_0^1\int_0^1 D^{-1}v_{yy}w^2\,dx\,dy\right|$$

$$+\max|D^{-1}v_y|\,\|D^{-1}w_y\|\,\|w_x\|$$

$$\leqslant c(\|D^{-1}w_y\|^2 + \|w_x\|^2)$$

where c depends on g and h from V_3. In sum,

$$\|w_x(t)\|^2 + \|D^{-1}w_y(t)\|^2 \leqslant \|w_x(0)\|^2 + \|D^{-1}w_y(0)\|^2$$

$$+ c\int_0^t (\|w_x(s)\|^2 + \|D^{-1}w_y(s)\|^2)\,ds.$$

Item 3. Differentiate (9) twice in x and multiply by w_{xx} and then differentiate (9) in y and multiply by w_y. Combine and integrate in x and y and use (6) and (7) to estimate each term in

$$\frac{1}{2}\frac{\partial}{\partial t}\int_0^1\int_0^1 (w_{xx}^2 + w_y^2)\,dx\,dy.$$

Integrate by parts in x and y, and use (6) and (7) to estimate each term. This involves estimates similar to the following:

$$\left|\int_0^1\int_0^1 w_{xx}wv_{xxx}\,dx\,dy\right| \leqslant \|v_{xxx}\|\,\|w_{xx}\|\max|w| \leqslant c(\|w_{xx}\|^2 + \|w_y\|^2),$$

$$\left|\int_0^1\int_0^1 v_y w_x w_y\,dx\,dy\right| \leqslant |v_y|_4\,|w_x|_4\,\|w_y\| \leqslant c(\|w_{xx}\|^2 + \|w_y\|^2),$$

$$\left|\int_0^1\int_0^1 D^{-1}v_y(w_{xx}w_y + w_x w_{xy})\,dx\,dy\right| \leqslant \tfrac{1}{2}\left|\int_0^1\int_0^1 D^{-1}v_{yy}w_x^2\,dx\,dy\right|$$

$$+\max|D^{-1}v_y|\,\|w_{xx}\|\,\|w_y\|$$

$$\leqslant c(\|w_{xx}\|^2 + \|w_y\|^2).$$

Use (9) to find

$$\frac{1}{2}\frac{\partial}{\partial t}\int_0^1\int_0^1 (D^{-2}w_{yy})^2\,dx\,dy = +\frac{1}{2}\int_0^1\int_0^1 D^{-3}w_{yy}(w^2)_{yy}\,dx\,dy$$
$$+\int_0^1\int_0^1 D^{-3}w_{yy}(vw)_{yy}\,dx\,dy.$$

Integrate in t and bound using estimates similar to the following:

$$\left|\int_0^1\int_0^1 D^{-2}w_{yy}\left(\int^x w_y w_y\,dx'\right)dx\,dy\right|$$
$$\leqslant \max_y \sqrt{\int_2^1 w_y^2(y)\,dx'}\;\|D^{-2}w_{yy}\|\;\|w_y\| \leqslant c\,(\|w_y\|^2+\|D^{-2}w_{yy}\|^2),$$
$$\left|\int_0^1\int_0^1 D^{-2}w_{yy}\left(\int^x D^{-2}w_{yy}w_{xx}\,dx'\right)dx\,dy\right|$$
$$\leqslant \max_y \sqrt{\int_0^1 w_{xx}^2\,dx'}\;\|D^{-2}w_{yy}\|^2 \leqslant c\,\|D^{-2}w_{yy}\|^2,$$
$$\left|\int_0^1\int_0^1 D^{-2}w_{yy}D^{-1}(w_x D^{-1}v_{yy})\,dx\,dy\right|$$
$$\leqslant \max_y \sqrt{\int_0^1 w_x^2(y)\,dx}\int_0^1\int_0^1 |D^{-2}w_{yy}(x,y)|\sqrt{\int_0^1 D^{-1}v_{yy}^2\,dx'}\,dx\,dy$$
$$\leqslant (\|w_{xx}\|\;\|w_y\|+\|w_x\|^2)^{1/2}\;\|D^{-2}w_{yy}\|\;\|D^{-1}v_{yy}\|$$
$$\leqslant c(\|w_{xx}\|^2+\|w_y\|^2+\|D^{-2}w_{yy}\|^2),$$

where we have used (6), (7), and (4). Combine to obtain

$$\|w_{xx}(t)\|^2+\|w_y(t)\|^2+\|D^{-2}w_{yy}(t)\|^2$$
$$\leqslant \|w_{xx}(0)\|^2+\|w_y(0)\|^2$$
$$+\|D^{-2}w_{yy}(0)\|^2$$
$$+c\int_0^t (\|w_{xx}(s)\|^2+\|w_y(s)\|^2+\|D^{-2}w_{yy}(s)\|^2)\,ds.$$

Apply Gronwall's lemma to items one to three to obtain (8). This completes the proof of the lemma.

Proof of the Theorem. We now establish the existence of a global solution to the initial value problem (1). We first prove the existence of a unique global solution u^η to the initial value problem

$$\frac{\partial u}{\partial t} + \eta(u_{xxxx} - u_{yy}) = D^{-1}u_{yy} - uu_x - u_{xxx}, \qquad 0 < \eta < 1 \tag{10}$$

with $\int_0^1 u(x, y, t)\,dx = 0$ and then show that the solution u^η converges to a solution of (1).

Item 1. Consider η-dependent initial data $g^\eta \in V_3$ converging to g in V_3. To establish the solvability of (10) we use Galerkin's method. Let the sequence $\phi_l(x, y)$ denote a trignometric orthonormal basis for V_3 with $\int_0^1 \phi_l(x, y)\,dx = 0$. We consider approximate solutions $u^n(x, y, t)$ of (10) in the form

$$u^n(x, y, t) = \sum_{l=1}^{n} c_{ln}(t)\,\phi_l(x, y),$$

where the coefficients $c_{ln}(t)$ are determined from the ordinary differential equations

$$\int_0^1 \int_0^1 (u_t^n + \eta(u_{xxxx}^n - u_{yy}^n) + u^n u_x^n + u_{xxx}^n - D^{-1}u_{yy}^n)\,\phi_l\,dx\,dy = 0 \quad (l \geqslant 1) \tag{11}$$

with

$$c_{ln}(0) = \int_0^1 \int_n^1 g(x, y)\,\phi_l(x, y)\,dx\,dy.$$

Equation (11) has a unique solution for all time. Since (11) depends analytically on $c_{ln}(t)$ we need only to show that $|c_{ln}(t)|$ are bounded for all time. Multiply (11) by $c_{ln}(t)$ and sum over l from 1 to n obtain

$$\int_0^1 \int_0^1 (u_t^n + \eta(u_{xxxx}^n - u_{yy}^n) + u^n u_x^n + u_{xxx}^n - D^{-1}u_{yy}^n)\,u^n\,dx\,dy = 0.$$

Integrate by parts and then integrate in t to obtain

$$\begin{aligned} \tfrac{1}{2}\,\|u^n(t)\|^2 + \eta \int_0^t (\|u_{xx}^n(t)\|^2 + \|u_y^n(t)\|^2)\,dt \\ = \tfrac{1}{2}\|u^n(0)\|^2 \leqslant \|g\|^2. \end{aligned} \tag{12}$$

The result follows from

$$\|u^n(t)\|^2 = \sum^{n} c_{ln}^2(t).$$

Next we establish the convergence of $u^n(t)$ to a solution of (10). This requires the following bounds. Multiply (11) by a differentiable function $d_l(t)$ vanishing at $t=0$ and $t=T$, and sum over l to show that

$$\int_0^1 \int_0^1 (u_t^n + \eta(u_{xxxx}^n - u_{yy}^n) + u_{xxx}^n + u^n u_x^n - D^{-1} u_{yy}^n)\, \phi^n \, dx\, dy = 0, \quad (13)$$

where $\phi^n = \sum_{l=1}^n d_l(t)\, \phi_l(x, y)$. Since u_{xx}^n and $D^{-2} u_{yy}^n$ belong to the span $(\phi_1, \phi_2, ..., \phi_n)$ it follows from (13), integration in t, and standard estimate that

$$\tfrac{1}{2} \| u_x^n(t) \|^2 + \tfrac{1}{2} \| D^{-1} u_y^n(t) \|^2$$
$$+ \eta \int_0^t \| u_{xxx}^n(t) \|^2 + 2 \| u_{xy}^n(t) \|^2 + \| D^{-1} u_{yy}^n(t) \|^2 \, dt$$

can be estimated from above by

$$\left| \int_0^t \int_0^1 \int_0^1 u_{xx}^n u^n u_x^n \, dx\, dy\, dt \right|$$
$$\leqslant \int_0^t \| u_x^n \| \, | u_x^n |_4^2 \, dt$$
$$\leqslant \max_t \| u_x^n(t) \| \left(\int_0^t \| u_{xx}^n(t) \|^2 \, dt \right)^{3/4} \left(\int_0^t \| u_{xy}^n(t) \|^2 \, dt \right)^{1/4}$$
$$\leqslant c + \tfrac{1}{4} \max_t \| u_x^n(t) \|^2,$$

$$\left| \int_0^t \int_0^1 \int_0^1 D^{-2} u_{yy}^n u^n u_x^n \, dx\, dy\, dt \right|$$
$$\leqslant \max_t \| D^{-1} u_y^n(t) \| \left(\int_0^t \| u_y^n(t) \|^2 \, dt \right)^{3/4} \left(\int_0^t \| u_{xx}^n(t) \|^2 \, dt \right)^{1/4}$$
$$\leqslant c + \tfrac{1}{4} \max_t \| D^{-1} u_y^n \|^2,$$

and

$$\| u_y^n(0) \|^2 + \| D^{-1} u_y^n(0) \|^2 \leqslant \| g_x \|^2 + \| D^{-1} g_y \|^2,$$

where c is independent of n and we have used integration by parts, (6), (7), and previous bounds. This gives the bound

$$\| u_x^n(t) \|^2 + \| D^{-1} u_y^n(t) \|^2 \leqslant c, \quad (14)$$

where c is independent of n.

Integrate

$$\frac{\partial}{\partial t}\int_0^1\int_0^1 u^n(t)\,\phi_l\,dx\,dy$$

$$=\int_0^1\int_0^1 (D^{-1}u^n_{yy}\phi_l-u^n_{xxx}\phi_l-u^nu^n_x\phi_l-\eta u^n_y\phi_{ly}+u^n_{xx}\phi_{lxx})\,dx\,dy$$

in t from t to $t+\Delta t$ and estimate and use previous bounds to prove

$$|(u^n(t+\Delta t)-u^n(t),\phi_l)|\leqslant c_l\Delta t, \tag{15}$$

where c_l is independent of n. From (12) and (15), the functions $(u^n(t),\phi_l)$ for fixed l and $n\geqslant l$ are uniformly bounded and equicontinuous on $[0,T]$. Select a diagonal subsequence so that $(u^n(t),\phi_l)$ converges uniformly as n tends to infinity for any fixed l. It follows that $u^n(t)$ converges to $u(t)$ weakly in L_2 uniformly in t on $[0,T]$. For $u\in V_3$, an application of Fourier series shows that for $\varepsilon>0$ there exists N_ε basis functions $w_l(x,y)$ such that

$$\int_0^1\int_0^1 u^2\,dx\,dy\leqslant\sum_{l=1}^{N_\varepsilon}\left(\int_0^1\int_0^1 uw_l\,dx\,dy\right)^2+\varepsilon\int_0^1\int_0^1 (u_x^2+(D^{-1}u_y)^2)\,dx\,dy. \tag{16}$$

By (16), (14), and the weak convergence of $u^n(t)$ in L_2 uniformly in t on $[0,T]$, it appears that $u^n(t)$ converges to $u(t)$ strongly in L_2 uniformly in t on $[0,T]$. It follows that $u(t)$ is summable and satisfies

$$\int_0^T\int_0^1\int_0^1 (u\phi_t-\eta(u\phi_{xxxx}-u\phi_{yy})$$

$$+u\phi_{xxx}+\tfrac{1}{2}u^2\phi_x+uD^{-1}\phi_{yy})\,dx\,dy\,dt=0,$$

where ϕ is of the form $\sum_{j=1}^N d_l\phi_l$ with any N.

Next we show that the solution belongs to V_3. This requires more estimates. Since u^n_{xxxx}, u^n_{yy}, and $D^{-4}u^n_{yyyy}$ belong to the span$(\phi_1,\phi_2,\dots,\phi_n)$, it follows from (13), integration in t and standard estimates that

$$\tfrac{1}{2}(\|u^n_{xx}\|^2+\|u^n_y\|^2+\|D^{-2}u^n_{yy}\|^2)$$

$$+\eta\int_0^t(\|u^n_{xxxx}\|^2+2\,\|u^n_{xxy}\|^2+2\,\|u^n_{yy}\|^2+\|D^{-2}u^n_{yyy}\|^2)\,dt$$

is estimated from above by

$$\left|\int_0^t\int_0^1\int_0^1 u^n_{xxxx}u^n u^n_x\,dx\,dy\,dt\right|$$
$$\leqslant 3\int_0^t |u^n_{xx}|_4^2\,\|u^n_x\|\,dt\leqslant c,$$
$$\left|\int_0^t\int_0^1\int_0^1 u^n_{yy}u^n u^n_x\,dx\,dy\,dt\right|$$
$$\leqslant \int_0^t\int_0^1\int_0^1 (|u^n_y u^n u^n_{xy}|+|u^n_y u^n_y u^n_x|)\,dx\,dy\,dt$$
$$\leqslant \max_t\|u^n_y(t)\|\left(\int_0^t\|u^n_{xy}\|^2\,dt\right)^{1/2}\left(\int_0^t\|u^n_{xx}\|^2\,dt\right)^{1/4}\left(\int_0^t\|u^n_y\|^2\,dt\right)^{1/4}$$
$$+\max_t\|u^n_y(t)\|\left(\int_0^t\|u^n_y\|^2\,dt\right)^{1/2}\left(\int_0^t\|u^n_{xxx}\|^2\,dt\right)^{1/4}\left(\int_0^t\|u_{xy}\|^2\,dt\right)^{1/4}$$
$$\leqslant c+\tfrac{1}{4}\max_t\|u^n_y\|^2,$$
$$\left|\int_0^t\int_0^1\int_0^1 D^{-4}u^n_{yyyy}u^n u^n_x\,dx\,dy\,dt\right|$$
$$\leqslant c\int_0^t\|D^{-2}u^n_{yy}\|^2\max|u^n_x|\,dt$$
$$+c\int_0^t\|D^{-2}u^n_{yy}\|\,\|D^{-2}u^n_{yy}\|\max_y\|u^n_{xx}\|\,dt$$
$$\leqslant c\,\tfrac{1}{4}\max_t\|D^{-2}u^n_{yy}\|^2+c,$$
$$\|u^n_{xx}(0)\|^2+\|u^n_y(0)\|^2+\|D^{-2}u^n_{yy}(0)\|^2$$
$$\leqslant\|g_{xx}\|^2+\|g_y\|^2+\|D^{-2}g_{yy}\|^2,$$

where we have used (6), (7), (4), and previous bounds. This implies

$$\|u^n_{xx}(t)\|^2+\|u^n_y(t)\|^2+\|D^{-2}u^n_{yy}(t)\|^2\leqslant c, \tag{17}$$

where c is independent of n.

Use (13) and the fact that u^n_{xxxxxx}, u^n_{xxyy}, $D^{-2}u^n_{yyyy}$, and $D^{-6}u^n_{yyyyyy}$ belongs to span$(\phi_1, \phi_2,..., \phi_n)$, to confirm that

$$\tfrac{1}{2}(\|u^n_{xxx}(t)\|^2+\|u^n_{xy}(t)\|^2+\|D^{-1}u^n_{yy}(t)\|^2+\|D^{-3}u^n_{yyy}(t)\|^2)$$
$$+\eta\int_0^t\|u^n_{xxxxx}(t)\|^2+2\,\|u^n_{xxxy}(t)\|^2+2\,\|u^n_{xyy}(t)\|^2$$
$$+2\,\|D^{-1}u^n_{yyy}(t)\|^2+\|D^{-3}u^n_{yyyy}\|^2$$

may be estimated from above by terms like

$$\left|\int_0^t \int_0^1 \int_0^1 u^n_{xxyy} u^n u^n_x \, dx\, dy\, dt\right|$$

$$\leqslant c \int_0^t \| u^n_{xy} \|^2 \max | u^n_x | + | u^n_y |_4^2 \, \| u^n_{xxx} \| \, dt$$

$$\leqslant c + \tfrac{1}{4} \max_t \| u^n_{xy} \|^2,$$

$$\left|\int_0^t \int_0^1 \int_0^1 D^{-2} u^n_{yyy} \, u^n u^n_x \, dx\, dy\, dt\right|$$

$$\leqslant c \int_0^t \| D^{-1} u^n_{yy} \|^2 \, \| u^n_{xxx} \|^{1/2} \, \| u^n_{xy} \|^{1/2}$$

$$+ c \int_0^t \| D^{-1} u^n_{yy} \| \, \| u^n_{xy} \|^{3/2} \, \| D^{-1} u^n_{yy} \|^{1/2}$$

$$\leqslant c + \tfrac{1}{8} \max_t \| D^{-1} u^n_{yy} \|^2,$$

$$\| u^n_{xxx}(0) \|^2 + \| u^n_{xy}(0) \|^2 + \| D^{-1} u^n_{yy}(0) \|^2 + \| D^{-3} u^n_{yyy}(0) \|^2$$

$$\leqslant \| g_{xxx} \|^2 + \| g_{xy} \|^2 + \| D^{-1} g_{yy} \|^2 + \| D^{-3} g_{yyy} \|^2,$$

where we have used (6), (7), (4), and previous bounds. This gives

$$\| u^n_{xxx}(t) \|^2 + \| u^n_{xy}(t) \|^2 + \| D^{-1} u^n_{yy}(t) \|^2 + \| D^{-3} u^n_{yyy}(t) \|^2 \leqslant c, \tag{18}$$

where c is independent of n.

The solution of (10) has the additional regularity $u \in L^\infty([0, T], V_3)$ and $u_t \in L^\infty([0, T]\ L_2)$. Let ψ be an infinitely differentiable function of period 1 in x and y, and vanishing at $t = 0$ and $t = T$, and consider the product $(u^n(t), \psi(t))$ where $(\,,)$ is the natural pairing of V_3 with V_3', the dual of V_3. Use (12-18) to confirm

$$\left| \int_0^T (u^n(t), \psi(t))\, dt \right| \leqslant \max_t \| u^n(t) \|_{V_3} \int_0^T \| \psi(t) \|_{V_3'} \leqslant c \int_0^T \| \psi(t) \|_{V_3'} \, dt.$$

The convergence of $u^n(t)$ to $u(t)$ in L_2 uniformly in t implies that $\int_0^T (u^n(t), \psi(t))\, dt$ to $\int_0^T (u(t), \psi(t))\, dt$. By the previous estimate $u \in L^\infty([0, T], V_3)$. Multiply (11) by $dc_{ln}(t)/dt$, sum over l, and use (12)–(18) to confirm $\| u^n_t(t) \| \leqslant c$, the constant being independent of n. To show that $u_t \in L^\infty([0, T]\ L_2)$, use $\|u^n_t(t)\| \leqslant c$ and the estimate

$$\left| \int_0^T (u^n_t(t), \psi(t))\, dt \right| = \left| \int_0^T (u^n(t), \psi_t(t))\, dt \right| \leqslant c \int_0^T \|\psi(t)\|\, dt.$$

The rest is as before.

That there is at most one solution of the initial value problem (10) follows as in the proof of Lemma 2.

Combine (6), previous bounds, the convergence of $u^n(t)$ to $u(t)$ in L_2 uniformly in t with $\int_0^1 u^n(x, y, t)\, dx = 0$ to establish that $\int_0^1 u(x, y, t)\, dx = 0$.

We have established the existence of a unique global solution $u^\eta(x, y, t)$ of (10) with $u^\eta \in L^\infty([0, T], V_3)$ and $u_t^\eta \in L^\infty([0, T], L_2)$.

Item 2. We prove that $u^\eta(x, y, t)$ converges to a solution of (1). This depends upon the uniform boundedness of u^η.

(a) Begin with a solution $u(x, y, t)$ of class V_3, multiply (10) by u, and integrate in x, y and then in t to obtain

$$\tfrac{1}{2} \| u(t)\|^2 + \eta \int_0^t (\| u_{xx} \|^2 + \| u_y \|^2)\, dt = \tfrac{1}{2} \| g \|^2 \tag{19}$$

and

$$\| u(t)\|^2 \leqslant \| g \|^2.$$

(b) Write (10) as

$$u_t + \eta(u_{xxxx} - u_{yy}) = D(D^{-2}u_{yy} - \tfrac{1}{2}u^2 - u_{xx}),$$

multiply by $D^{-2}u_{yy} - \frac{1}{2}u^2 - u_{xx}$, integrate in x, y, and t, and integrate by parts to confirm

$$\begin{aligned} &\tfrac{1}{2} \| u_x \|^2 + \tfrac{1}{2} \| D^{-1}u_y \|^2 \\ &\qquad + \eta \int_0^t (\| u_{xxx} \|^2 + 2 \| u_{xy} \|^2 + \| D^{-1}u_{yy} \|^2)\, dt \\ &\leqslant \tfrac{1}{2} \| g_x \|^2 + \tfrac{1}{2} \| D^{-1}g_y \|^2 \\ &\qquad + \left| \int_0^1 \int_0^1 u^3\, dx\, dy \right| + \eta \int_0^t \int_0^1 \int_0^1 |u_{xxxx} - u_{yy}|\, u^2\, dx\, dy\, dt. \end{aligned}$$

Transform the right-hand side as follows:

$$\begin{aligned} &\left| \int_0^1 \int_0^1 u^3\, dx\, dy \right| \\ &\qquad \leqslant \| g \| \, \| u_x \|^{3/2} \, \| D^{-1}u_y \|^{1/2}, \end{aligned}$$

$$\left|\int_0^t\int_0^1\int_0^1 u_{xxxx}u^2\,dx\,dy\,dt\right|$$
$$\leqslant \int_0^t\int_0^1\int_0^1 |u_{xx}u_{xx}u|\,dx\,dy\,dt$$
$$\leqslant \int_0^t \|u\|\,\|u_{xxx}\|^{3/2}\,\|u_{xx}\|^{1/2}\,dt$$
$$\leqslant \|g\|\left(\int_0^t \|u_{xxx}\|^2\,dt\right)^{3/4}\left(\int_0^t \|u_{xy}\|^2\,dt\right)^{1/4},$$

$$\left|\int_0^t\int_0^1\int_0^1 u_{yy}u^2\,dx\,dy\,dt\right|$$
$$\leqslant \int_0^t\int_0^1\int_0^1 |D^{-1}u_{yy}uu_x|\,dx\,dy\,dt$$
$$\leqslant \|g\|\int_0^t \|D^{-1}u_{yy}\|\max|u_x|\,dt$$
$$\leqslant \|g\|\int_0^t \|D^{-1}u_{yy}\|\,\|u_{xxx}\|^{1/2}\,\|u_{xy}\|^{1/2}\,dt$$
$$\leqslant \|g\|\left(\int_0^t \|D^{-1}u_{yy}\|^2\,dt\right)^{1/2}\left(\int_0^t \|u_{xxx}\|^2\,dt\right)^{1/4}\left(\int_0^t \|u_{xy}\|^2\,dt\right)^{1/4}.$$

Apply Young's inequality and proceed as in the proof of Lemma 1 to confirm that for small $\|g\|$,

$$\|u_x(t)\|^2+\|D^{-1}u_y(t)\|^2\leqslant c, \tag{20}$$

where c is independent of η and depends only on $\|g_x\|$, $\|D^{-1}g_y\|$.

(c) Let $G_5(u)$ denote the gradient of $F_5(u)$. Multiply (10) by $G_5(u)$, integrate in x and y, and use $(G_5(u),\,D^{-1}u_{yy}-uu_x-u_{xxx})=0$ to confirm

$$\frac{\partial}{\partial t}F_5(u)+\eta\int_0^1\int_0^1 (u_{xxxx}-u_{yy})\,G_5(u)\,dx\,dy=0. \tag{21}$$

Consider only the linear terms of $G_5(u)$. Then

$$\eta\int_0^1\int_0^1 (u_{xxxx}-u_{yy})(u_{xxxx}-u_{yy}+D^{-4}u_{yyyy})\,dx\,dy$$
$$=\eta\int_0^1\int_0^1 (u_{xxxx}^2+2u_{xxy}^2+2u_{yy}^2+(D^{-2}u_{yyy})^2)\,dx\,dy.$$

Use (6), (7), Young's inequality and previous bounds to estimate the quadratic and cubic terms of $G_5(u)$ in the second integral of (21). These estimates are similar to the following:

$$\left|\eta \int_0^t \int_0^1 \int_0^1 (u_{xxx} - D^{-1}u_{yy})\, D^{-1}(u^2)_{yy}\, dx\, dy\, dt\right|$$

$$\leqslant 2\eta \int_0^t (\| u_{xxx} \| + \| D^{-1}u_{yy} \|)$$

$$\times \left(\| u_{xx} \| \max_y \sqrt{\int_0^1 u^2\, dx} + \| u_y \| \max_y \sqrt{\int_0^1 u_y^2\, dx} \right)$$

$$\leqslant c + \varepsilon \max_t \| u_y \|^2,$$

$$\left|\eta \int_0^t \int_0^1 \int_0^1 (u_{xxxx} - u_{yy})(D^{-1}u_y)^2\, dx\, dy\, dt\right|$$

$$\leqslant \eta \int_0^t (\| u_{xxxx} \| + \| u_{yy} \|)\, \| D^{-1}u_y \| \max |D^{-1}u_y|$$

$$\leqslant c + \varepsilon\eta \int_0^t (\| u_{xxxx} \|^2 + \| u_{yy} \|^2)\, dt,$$

$$\left|\eta \int_0^t \int_0^1 \int_0^1 (u_{xxx} - D^{-1}u_{yy})\, uD^{-1}u_{yy}\, dx\, dy\, dt\right|$$

$$\leqslant \eta \int_0^t (\| u_{xxx} \| + \| D^{-1}u_{yy} \|)\, \| D^{-1}u_{yy} \| \max |u|$$

$$\leqslant c + \varepsilon(\max_t \| u_y \|^2 + \max_t \| u_{xx} \|^2),$$

$$\left|\eta \int_0^t \int_0^1 \int_0^1 (u_{xxxx} - u_{yy})(u_{xx}u + u_x^2)\, dx\, dy\, dt\right|$$

$$\leqslant c + \varepsilon\eta \int_0^t \| u_{xxxx} \|^2 + \| u_{yy} \|^2\, dt,$$

$$\left|\eta \int_0^t \int_0^1 \int_0^1 (u_{xxxx} - u_{yy})\, uD^{-2}u_{yy}\, dx\, dy\, dt\right|$$

$$\leqslant \eta \int_0^t \| u_{xxxx} \|\, |u|_4\, |D^{-2}u_{yy}|_4\, dt + \eta \int_0^t \| u_{yy} \|\, |u|_4\, |D^{-2}u_{yy}|_4\, dt$$

$$\leqslant c + \varepsilon\eta \int_0^t (\|u_{xxxx}\|^2 + \| D^{-2}u_{yyy} \|^2)\, dt.$$

where c is independent of η, ε is arbitrary, and we have used (6), (7), and Young's inequality and previous bounds. Now integrate (21) in t and use the estimates of Item 3 of Lemma 1 and the preceding estimate, and select ε to obtain

$$\| u_{xx}(t)\|^2 + \| u_y(t)\|^2 + \| D^{-2}u_{yy}(t)\|^2 \leqslant c, \tag{22}$$

where c is independent of η and depends only on $\| g_{xx} \|$, $\| g_y \|$, and $\| D^{-2} g_{yy} \|$.

(d) Use $F_7(u)$ of Item 4 in the proof of Lemma 1, compute its gradient $G_7(u)$, and proceed as in part (c) to prove that

$$\| u_{xxx}(t) \|^2 + \| u_{xy}(t) \|^2 + \| D^{-1} u_{yy}(t) \|^2 + \| D^{-3} u_{yyy}(t) \|^2 \leqslant c, \tag{23}$$

where c is independent of η.

By (19)–(23), u^η is bounded in the norm of V_3 by a constant which depends only on $\| g^\eta \|_{V_3}$. Since g^η converges to g in V_3, g^η is bounded independently of η in V_3. Thus the preceding bound is independent of η. The estimate $\| u_t^\eta \| \leqslant c$, with c independent of η, follows directly from the equation (10). We now discuss the converge of $u^\eta(x, y, t)$ to a solution $u(x, y, t)$ of (1). By the uniform bounds $\| u^\eta(t) \|$ and $\| u_t^\eta(t) \|$, we may select a subsequence $u^{\eta'}(t)$ which converges weakly in L_2, uniformly in t on any bounded interval. Using the bounds for $\| u_x^\eta \|$ and $\| D^{-1} u_y^\eta \|$ and (16), it follows that $u^{\eta'}(t)$ converges strongly in L_2 uniformly in t on bounded intervals. Multiply (10) by a function $\psi(x, y, t)$ which is infinitely differentiable, and periodic in x and y, and vanishes at $t = 0$ and $t = T$, and use (19)–(23) to obtain

$$\int_0^T \int_0^1 \int_0^1 u^\eta \phi_t + u^\eta D^{-1} \phi_y + \tfrac{1}{2} u^{\eta^2} \phi_x - u^\eta \phi_{xxx} \, dx \, dy \, dt = 0, \tag{24}$$

up to an error, which tends to zero with η. By the strong converges of $u^{\eta'}$ to u in L_2, uniformly in t on $[0, T]$, and by (24), we find that u is a weak solution of (10): u is locally summable and satisfies

$$\int_0^T \int_0^1 \int_0^1 (u\phi_t + uD^{-1}\phi_y + \tfrac{1}{2} u^2 \phi_x - u\phi_{xxx}) \, dx \, dy \, dt = 0.$$

Now use (19)–(23) and proceed as in the proof of Item 1 to show that $u \in L^\infty([0, T], V_3)$ and $u_t \in L^\infty([0, T], L_2)$ and $\int_0^1 u(x, y, t) \, dx = 0$. It follows that u solves (1) and that $u(x, y, 0) = g(x, y)$, since $u^\eta = g^\eta \to g$ strongly in V_3. Since u is uniquely determined by the data g, it follows that the full sequence u^η converges to the same solution u of (1).

The bounds of Lemma 1 show that this solution exists for all time. Lemma 3 shows that $g \in V_3 \to u(t) \in V_2$ is continuous uniformly in t on bounded intervals. This completes the proof of the theorem.

References

1. V. E. Zakharov and E. I. Schulman, *Physica D* **1** (1980), 192.
2. J. E. Lin and H. H. Chen, *Phys. Lett. A* **89** (1982), 163. H. H. Chen, Y. C. Lee, and J. E. Lin, *Physica D* **9** (1983), 439.
3. A. S. Fokas and P. M. Santini, Recursion operators and bihamiltonian structures in multidimensions, preprint.
4. I. M. Krichever and S. P. Novikov, "The Inverse Problem Method and Holomorphic Bundles on Riemann Surfaces," Mathematical Physics Reviews I, Chap. I, pp. 5–26, 1980.